Association of measured data to geometrical features

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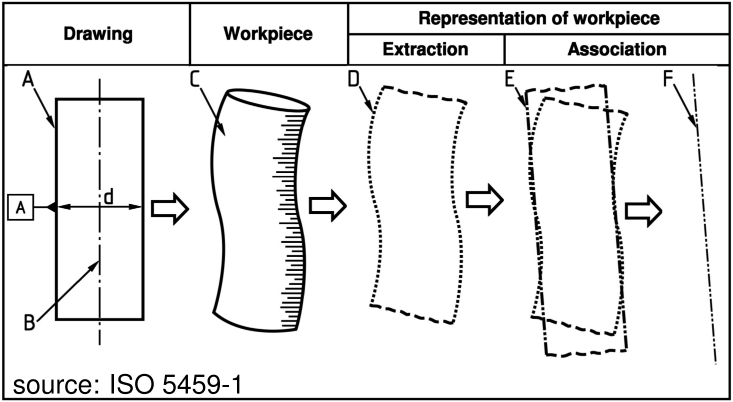
The task of the association operation



The task of the association operation is to fit a nominal geometrical feature, which has been the design intend, to the measured data points, by using a suitable association criterion and a proper algorithm.

Practically relevant geometrical features

In precision engineering the following geometrical features are of practical relevance:

- geometrical features in 2D

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  - cone
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Beside it there are other important geometrical features in 2D and 3D, such as helical lines, threads, gear geometries, turbine blades, free form surfaces of any kind, etc.

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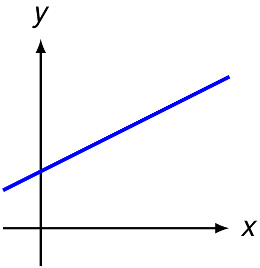
In the following suitable parametrisations for the most relevant cases will be given.

Parametrisation of a straight line in 2D

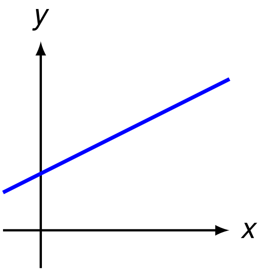


In order to get a parametrisation of a straight line, we use the formula

$$d = \mathbf{p} \cdot \mathbf{n} - p. \tag{1}$$

for the distance of a point from a straight line. All points on the straight line must obviously have the distance zero.

Parametrisation of a straight line in 2D



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Expressing the normal vector $\mathbf{n}$ by polar coordinates, we obtain

$$y \cos\alpha - x \sin\alpha - p = 0. \tag{2}$$

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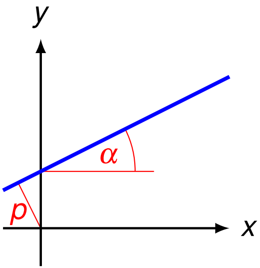

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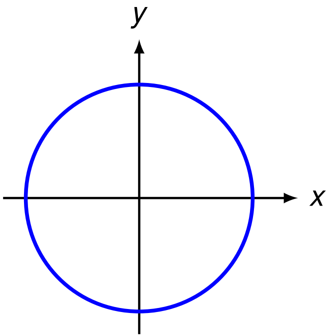
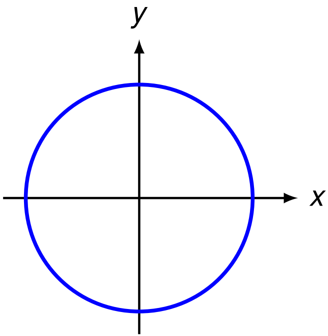
The straight line in 2D has the two parameters α and p.

Parametrisation of a circle in 2D



A circle is defined to be the locus of all points having a constant distance r, called the radius of the circle, from a fixed point $\mathbf{P}_0$, called the centre of the circle.

Parametrisation of a circle in 2D



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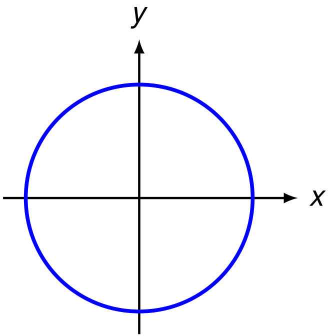
Using the formula for the distance of two points in 2D for the case $d = r$, we obtain

$$\|\mathbf{p} - \mathbf{p}_0\| - r = 0. \qquad (3)$$

Parametrisation of a circle in 2D



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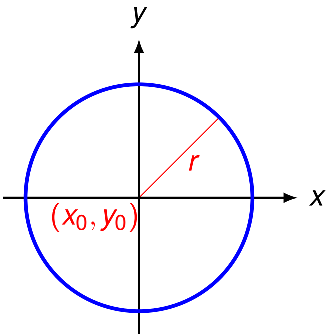
Since in 2D we have $\mathbf{p} = (x, y)^T$ and $\mathbf{p}_0 = (x_0, y_0)^T$, we obtain from the definition of the 2D norm

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} - r = 0. \tag{4}$$

Parametrisation of a circle in 2D



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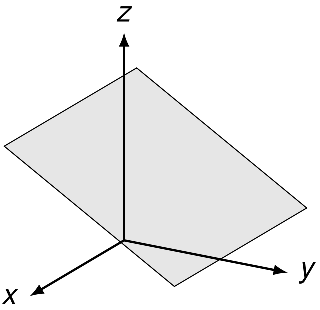
The circle in 2D has the three parameters x_0, y_0 and r.

Parametrisation of a plane

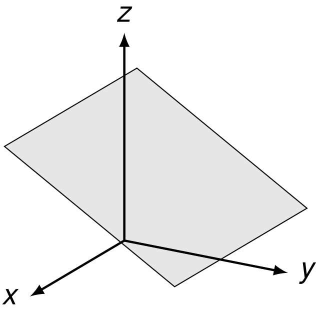


In order to get a parametrisation of a plane, we use the formula

$$d = \mathbf{p} \cdot \mathbf{n} - p. \qquad (5)$$

for the distance of a point from a plane. All points in the plane must obviously have the distance zero.

Parametrisation of a plane



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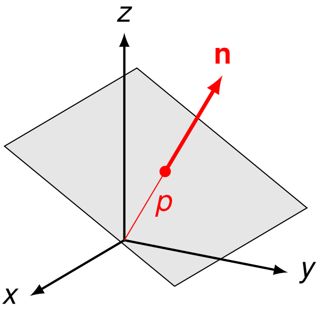
Expressing the normal vector **n** by spherical coordinates, we obtain

$$(x\cos\varphi + y\sin\varphi)\sin\vartheta + z\cos\vartheta - p = 0. \tag{6}$$

Parametrisation of a plane



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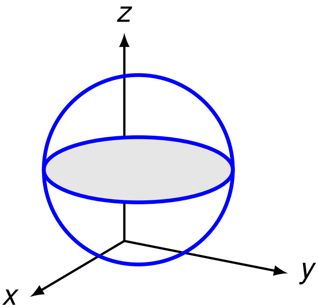
Expressing the normal vector **n** by spherical coordinates, we obtain

$$(x \cos \varphi + y \sin \varphi) \sin \vartheta + z \cos \vartheta - p = 0. \tag{6}$$

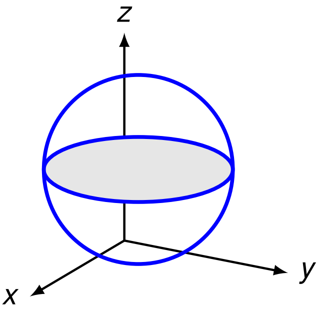
The plane has the three parameters ϑ, φ and p.

Parametrisation of a sphere



A sphere is defined to be the locus of all points having a constant distance r, called the radius of the sphere, from a fixed point $\mathbf{P}_0$, called the centre of the sphere.

Parametrisation of a sphere



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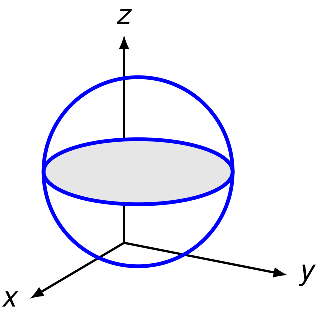
Using the formula for the distance of two points in 3D for the case $d = r$, we obtain

$$\|\mathbf{p} - \mathbf{p}_0\| - r = 0. \qquad (7)$$

Parametrisation of a sphere



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Since in 3D we have $\mathbf{p} = (x, y, z)^T$ and $\mathbf{p}_0 = (x_0, y_0, z_0)^T$, we obtain from the definition of the 3D norm

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} - r = 0. \tag{8}$$

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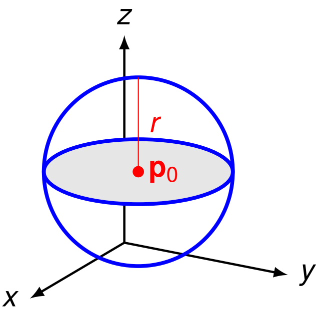

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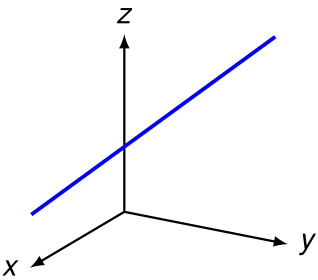
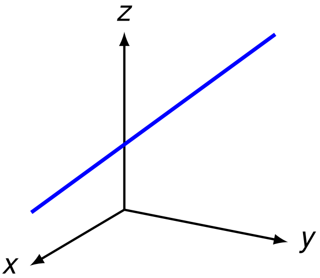
The sphere has the four parameters x_0, y_0, z_0 and r.

Parametrisation of a straight line in 3D



In order to parametrise the straight line in 3D, we use the formulae for the distance of a point from a straight line in 3D. For all points on the straight line, the distance must be zero.

Parametrisation of a straight line in 3D



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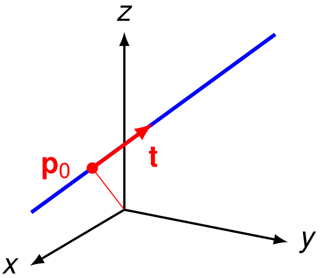
For the case $d = 0$ we obtain from the formulae for the distance of a point from a straight line in 3D

$$\|(\mathbf{p} - \mathbf{p}_0) \times \mathbf{t}\| = 0, \tag{9}$$

with the additional requirement

$$\mathbf{p}_0 \cdot \mathbf{t} = 0. \tag{10}$$

Parametrisation of a straight line in 3D



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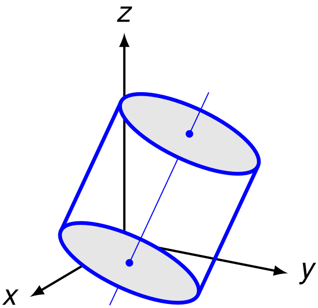
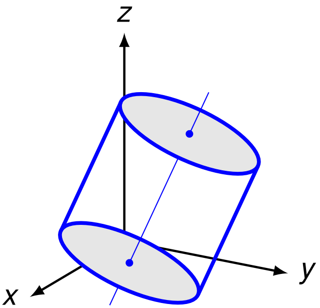
The straight line in 3D has four parameters.

Parametrisation of a cylinder



A cylinder is defined to be the locus of all points having a constant distance r, called the radius of the cylinder, from a fixed straight line, called the axis of the cylinder.

Parametrisation of a cylinder



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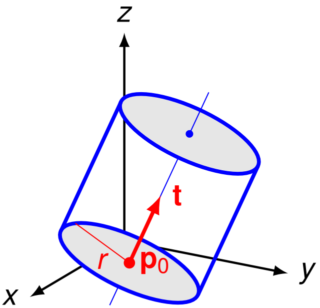
For the case $d = r$ we obtain from the formulae for the distance of a point from a straight line in 3D

$$\|(\mathbf{p} - \mathbf{p}_0) \times \mathbf{t}\| - r = 0, \qquad (11)$$

with the additional requirement

$$\mathbf{p}_0 \cdot \mathbf{t} = 0. \qquad (12)$$

Parametrisation of a cylinder



A cylinder is defined to be the locus of all points having a constant distance r, called the radius of the cylinder, from a fixed straight line, called the axis of the cylinder.

For the case $d = r$ we obtain from the formulae for the distance of a point from a straight line in 3D

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with the additional requirement

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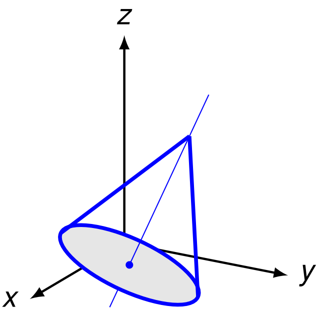
The cylinder has five parameters.

Parametrisation of a cone

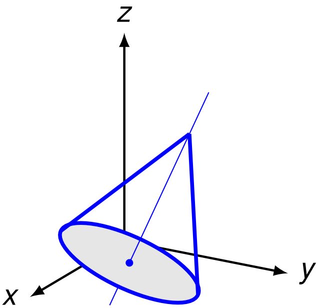


A cone is the locus of all points, for which the condition

$$\frac{r}{d} = \tan\frac{\beta}{2} \qquad (13)$$

is fulfilled, where r is the distance of a particular point from a straight line, the axis of the cone, d the distance from a fixed point $\mathbf{P}_0$, the apex of the cone, taken along the axis, and β the cone angle.

Parametrisation of a cone



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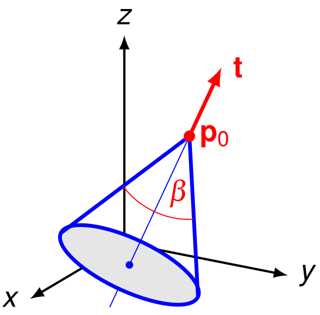
Using the formulae for the distance of a point from a straight line and a plane, respectively, we obtain

$$\|(\mathbf{p}-\mathbf{p}_0)\times\mathbf{t}\|\cos\frac{\beta}{2} - \big((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{t}\big)\sin\frac{\beta}{2} = 0. \qquad (14)$$

Parametrisation of a cone



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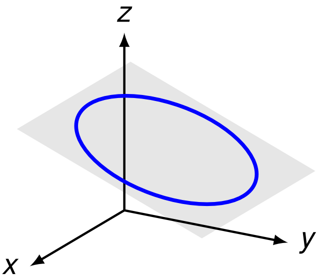
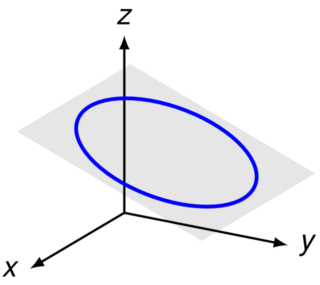
The cone has six parameters.

Parametrisation of a circle in 3D



A circle in 3D is the intersection of a cylinder and a plane perpendicular to the axis of this cylinder, i. e. it is the locus of all points, which have a constant distance r, the radius of the circle, from a straight line and zero distance from a plane perpendicular to this straight line through a fixed point $\mathbf{P}_0$, the centre of the circle.

Parametrisation of a circle in 3D



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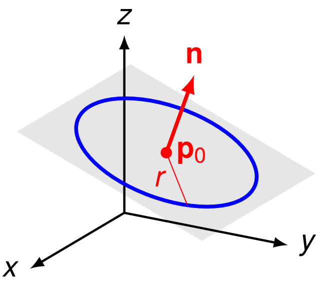
Using the formulae for the distance of a point from a straight line and a plane, respectively, we obtain by taking the square root of the sum of the squared distances

$$\sqrt{\left(\|(\mathbf{p}-\mathbf{p}_0)\times\mathbf{n}\|-r\right)^2+\left((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{n}\right)^2}=0. \qquad (15)$$

Parametrisation of a circle in 3D



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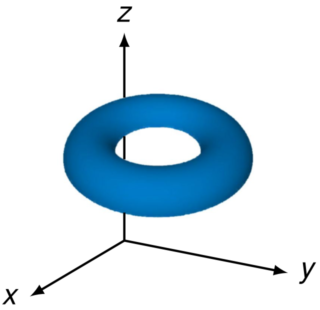
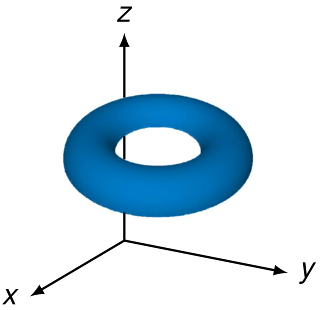
Parametrisation of a torus



A torus is defined to be the locus of all points, which have a fixed distance from a circle in 3D, which is called the small radius of the torus and will be denoted here by r, while the radius of the circle, which is called the large radius of the torus, will be denoted by R.

Parametrisation of a torus



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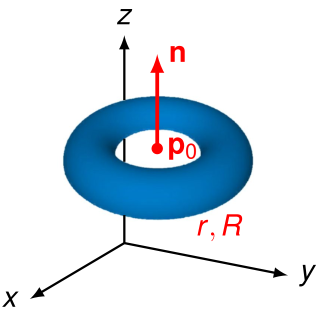
From the formula for the circle in 3D it can be deduced (for $r = 0$ the torus must obviously become a circle in 3D), that the formula for the torus is given by:

$$\sqrt{(\|(\mathbf{p} - \mathbf{p}_0) \times \mathbf{n}\| - R)^2 + ((\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n})^2} - r = 0. \qquad (16)$$

Parametrisation of a torus



A torus is defined to be the locus of all points, which have a fixed distance from a circle in 3D, which is called the small radius of the torus and will be denoted here by r, while the radius of the circle, which is called the large radius of the torus, will be denoted by R.

From the formula for the circle in 3D it can be deduced (for $r = 0$ the torus must obviously become a circle in 3D), that the formula for the torus is given by:

$$\sqrt{\left(\|(\mathbf{p}-\mathbf{p}_0)\times\mathbf{n}\| - R\right)^2 + \left((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{n}\right)^2} - r = 0. \qquad (16)$$

The torus has seven parameters.

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The reason for this requirements is, that large measurement errors make a kind of smoothing necessary, which can only be provided by the least squares criterion, but not by the Chebyshev criterion.

Local and global minima

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Objective functions of most practical problems are not convex!

The least squares criterion

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If the measured n data points are given by the location vectors $\mathbf{p}_i = (x_i, y_i)^T$ in 2D and $\mathbf{p}_i = (x_i, y_i, z_i)^T$ in 3D, respectively, and their respective distances are denoted by $d_i(\mathbf{p}_i, \mathbf{a})$, where $\mathbf{a}$ denotes the vector of the parameters, the LS criterion is described by

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The least squares criterion depends on all measured points, i. e. the parameters change, if one of the measured data points changes.

Solving LS problems (necessary condition)

The solution of least squares problems requires to minimise a function of the data and the parameters by determining an optimal parameter set. This function is called the objective function.

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A necessary condition for a minimum of the objective function is, that its gradient with respect to the parameters must be zero, i. e. if $f(a_1, \ldots, a_m)$ is the objective function of the parameter set $\{a_1, \ldots, a_m\}$,

$$\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial a_1} \\ \vdots \\ \dfrac{\partial f}{\partial a_m} \end{pmatrix} = 0 \tag{18}$$

must be fulfilled.

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The sufficient condition for a minimum of the objective function is, that its Hessian with respect to the parameters obtained by the necessary condition must be positive definite, i. e. the matrix

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial a_1 \partial a_1} & \cdots & \frac{\partial^2 f}{\partial a_1 \partial a_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial a_m \partial a_1} & \cdots & \frac{\partial^2 f}{\partial a_m \partial a_m} \end{pmatrix} \tag{19}$$

must be positive definite.

Example 1: Least squares straight line in 2D

The distance of the i-th measured point (x_i, y_i) from a straight line in 2D is given by

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where α is the slope angle of the straight line, and p the perpendicular distance from the origin of the coordinate system.

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Thus according to equation (30), we have to minimise the objective function

$$\eta(\alpha, p) = \sum_{i=1}^{n} (y_i \cos\alpha - x_i \sin\alpha - p)^2$$

by choosing suitable parameters α and p.

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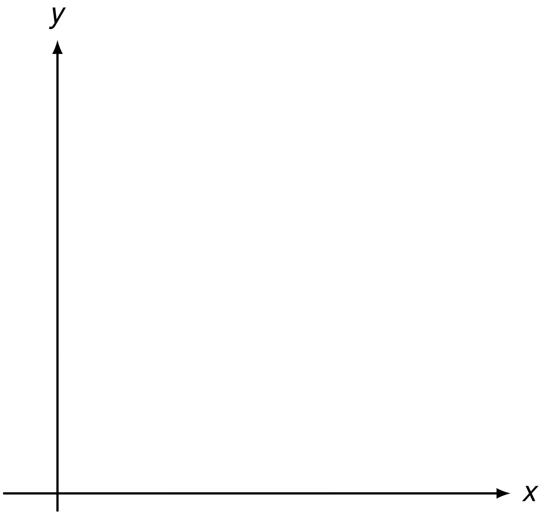

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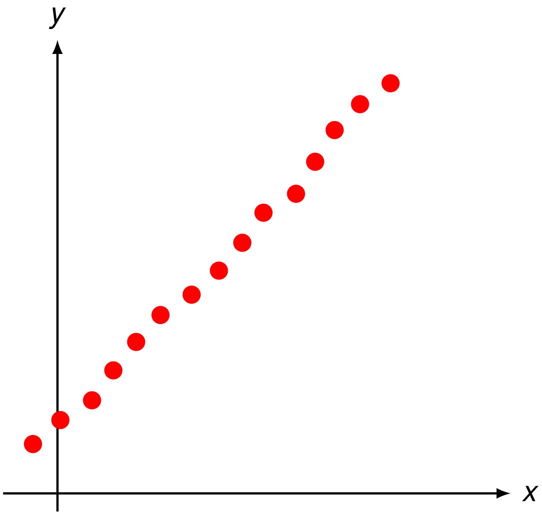

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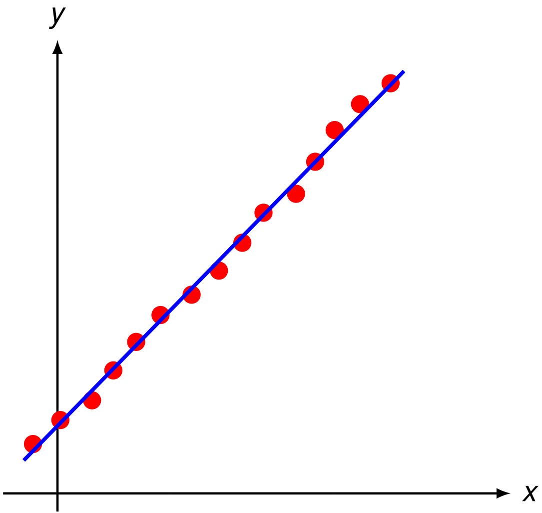

$\alpha = 45,65^\circ$
$p = 0,52$
$\eta_{\text{opt}} = 1,3$

Example 2: Least squares circle in 2D

The distance of the i-th measured point (x_i, y_i) from a circle in 2D is given by

$$d_i = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r,$$

where x_0 and y_0 are the centre coordinates, and r is the radius of the circle.

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Thus according to equation (30), we have to minimise the objective function

$$\eta(x_0, y_0, r) = \sum_{i=1}^{n} \left(\sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r \right)^2$$

by choosing suitable parameters x_0, y_0 and r.

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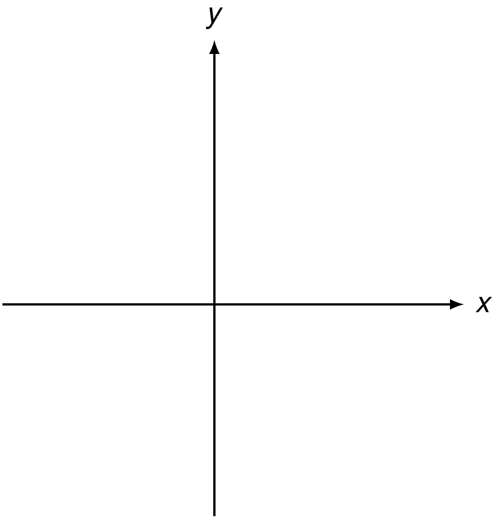

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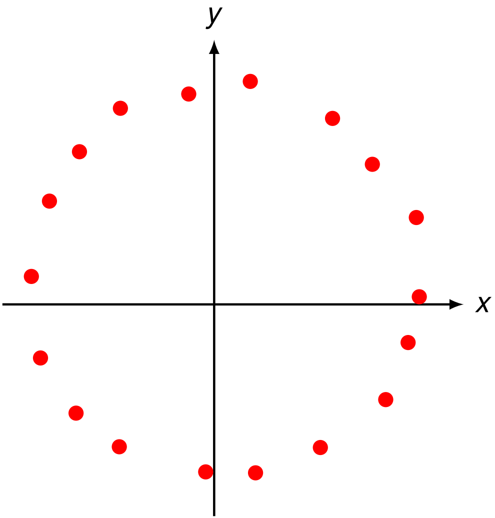

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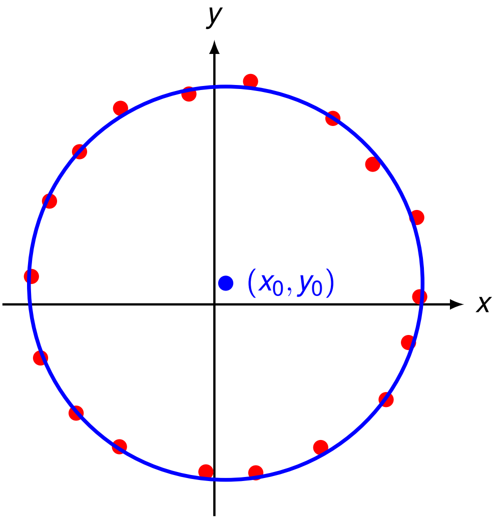

$x_0 = 0,15$
$y_0 = 0,28$
$r = 2.6$
$\eta_{\text{opt}} = 1,5$

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The Chebyshev criterion depends only on some of the measured points, i. e. the parameters do not necessarily change, if one of the measured data points changes.

The minimum zone criterion

If we set

$$t = \max_{i=1,\dots,n} |d_i(\mathbf{p}_i, \mathbf{a})| \,,$$

we have $|d_i| \leq t$ ($i = 1, \dots, n$), i. e. all measured data points are included within a zone of width $2t$.

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Equation (31) changes to

$$\min_{\mathbf{a}} t \tag{21}$$

under the constraints

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The application of the Chebyshev criterion is equivalent to the search for a minimum zone (MZ), which includes all measured data points.

Note that the zones must be closed or semi closed!

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Note that the geometrical features used by the single sided Chebyshev criteria must be closed or semi closed features!

Solving Chebyshev problems

There is a lot of different methods to solve Chebyshev problems, some are only heuristic, i. e. they have not theoretically be justified, some are more or less good approximations, and some are based on optimisation theory. Only the latter method will be considered here.

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In optimisation theory the problem is stated mathematically by

$$\min_{\mathbf{a}} f(\mathbf{a}) \tag{23}$$

under the constraints

$$g_k(\mathbf{a}) = 0, \qquad k \in E, \tag{24}$$

$$g_k(\mathbf{a}) \geq 0, \qquad k \in I, \tag{25}$$

where $f(\mathbf{a})$ is the objective function of the parameters $\mathbf{a}$, $g_k(\mathbf{a})$ the constraints, E and I the index sets of the equality and inequality constraints, respectively.

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where $\lambda_k > 0$ $(k \in E \cup I^*)$ are the so called Lagrange multipliers. A necessary condition for a minimum of the objective function is, that the gradient of the Lagrange function with respect to the parameters must be zero. This leads to the Kuhn-Tucker conditions

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Note that the Lagrange multipliers have to be positive!

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The sufficient condition for a minimum of the objective function is, that the quadratic form

$$\mathbf{s}^T \mathbf{H} \mathbf{s}$$

must be positive definite, where $\mathbf{s}$ is a feasible direction and $\mathbf{H}$ is the Hessian of the Lagrange function with respect to the parameters obtained by the necessary condition.

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- **Sufficient conditions**:
  The quadratic form $\mathbf{s}^T \mathbf{H} \mathbf{s}$ for a feasible direction $\mathbf{s}$ and the Hessian of the Lagrange function with respect to the optimal parameter set $\mathbf{a}^*$ must be positive definite.

Example 3: Minimum zone straight line in 2D

The distance of the i-th measured point (x_i, y_i) from a straight line in 2D is given by

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Using the Chebyshev criterion according to equations (32) and (33), we require to minimise the width $2t$ of a straight stripe by choosing suitable parameters α and p under the constraints

$$t > 0 \quad \text{and} \quad t - |y_i \cos\alpha - x_i \sin\alpha - p| \geq 0, \quad i = 1, \ldots, n.$$

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The application of the Chebyshev criterion to a straight line in 2D yields a straight stripe of minimum width, which includes all measured data points.

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If the single contacting point is projected onto the straight line through the other two contacting points, the projected point must strictly be located between these two points.

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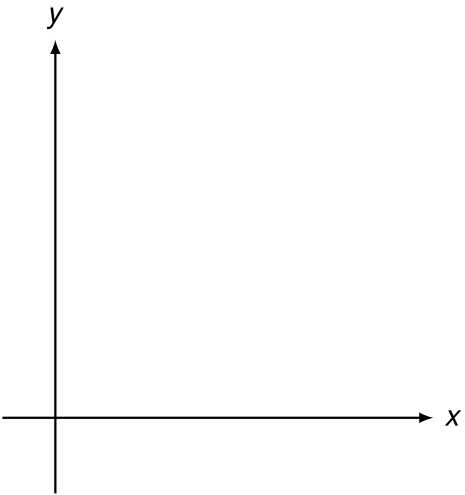

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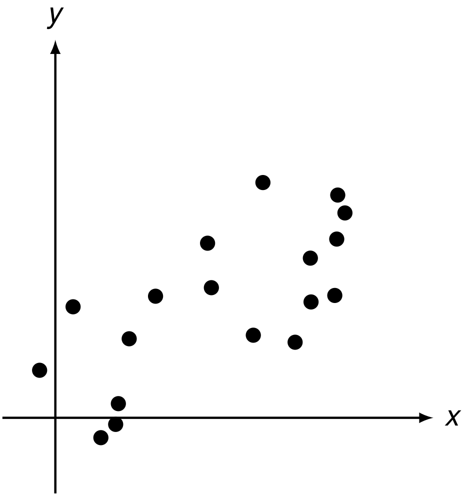

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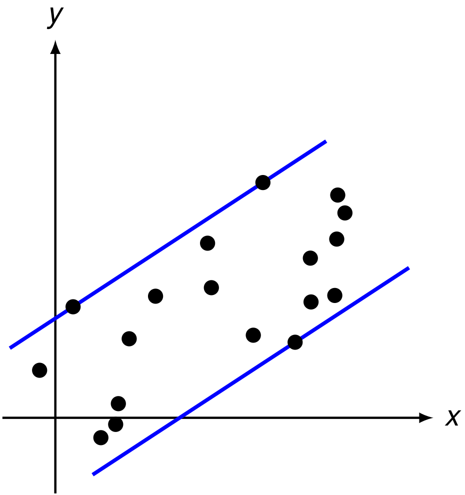

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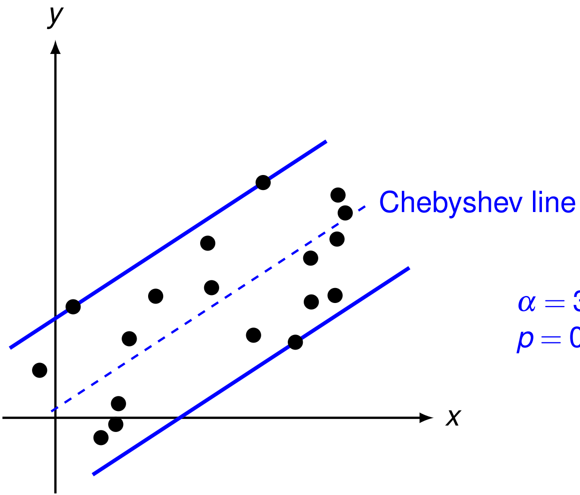

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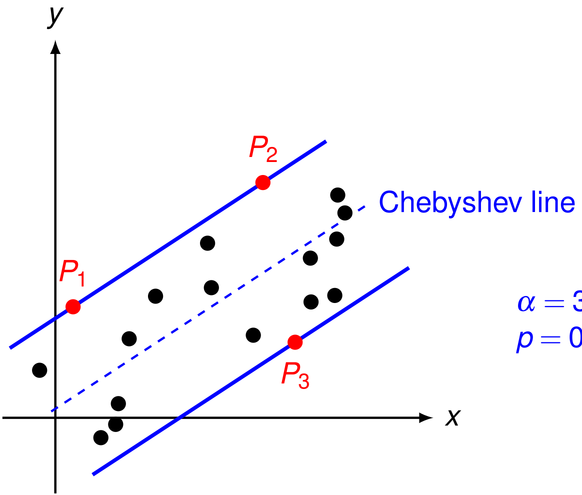

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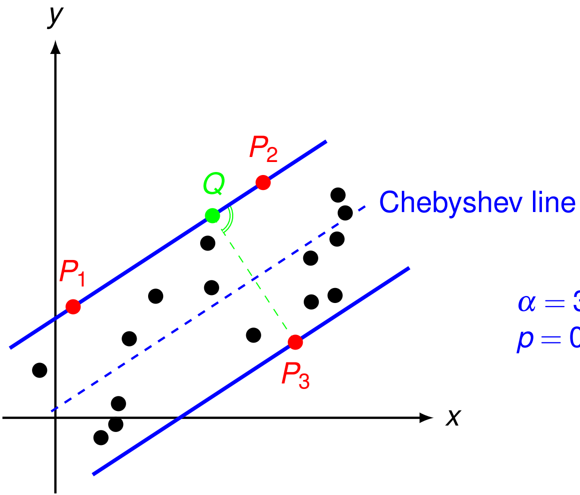

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Example 4: Minimum zone circle in 2D

The distance of the i-th measured point (x_i, y_i) from a circle in 2D is given by

$$d_i = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r,$$

where x_0 and y_0 are the centre coordinates, and r is the radius of the circle.

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Using the Chebyshev criterion according to equations (32) and (33), we require to minimise the width $2t$ of an annulus by choosing suitable parameters x_0, y_0 and r under the constraints

$$t > 0 \quad \text{and} \quad t - \left| \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r \right| \geq 0, \quad i = 1, \ldots, n.$$

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The application of the Chebyshev criterion to a circle in 2D yields an annulus of minimum width, which includes all measured data points.

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If degeneracy (more contacting points than necessary) is ignored, there are only two possibilities for a minimum of the minimum zone circle problem in 2D.

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The first possibility for an optimal solution of the minimum zone circle problem in 2D is:

There are four contacting points.

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- When the two contacting points on the inner boundary circle are radially projected on the outer boundary circle, the cord connecting these projected points must intersect the cord connecting the two contacting points on the outer boundary circle.

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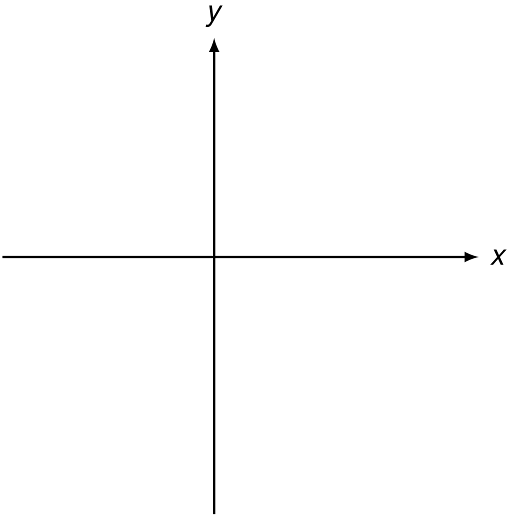

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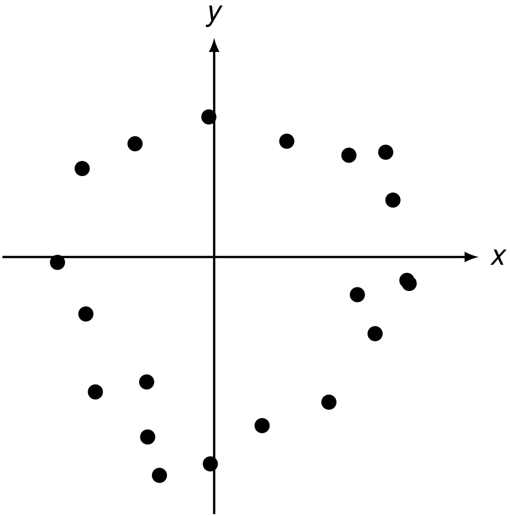

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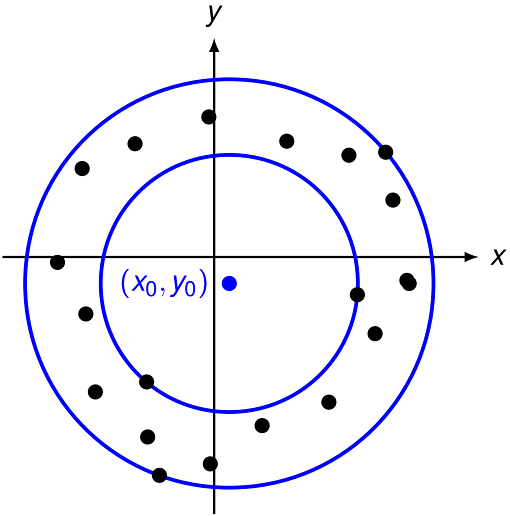

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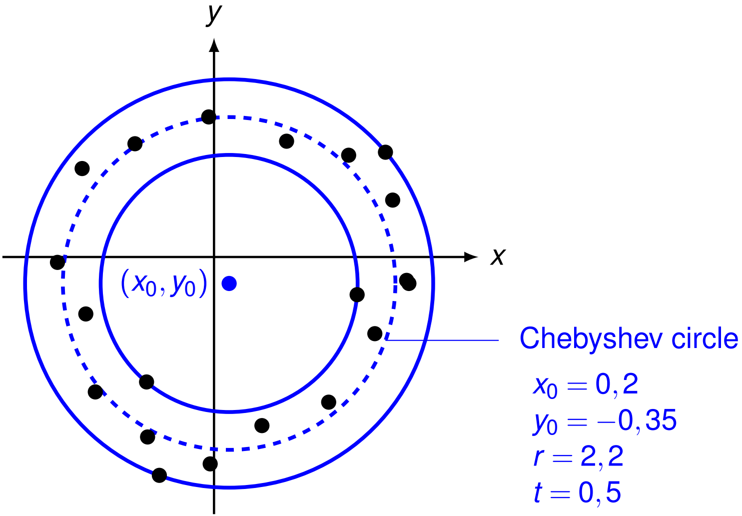

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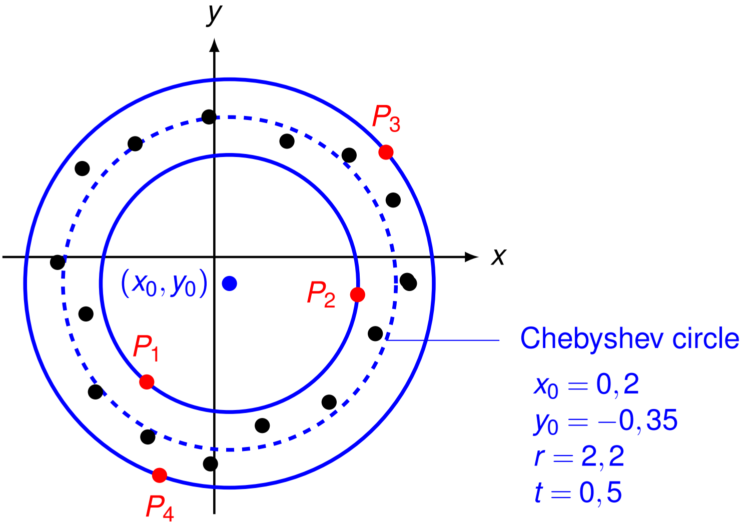

$x_0 = 0,2$
$y_0 = -0,35$
$r = 2,2$
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The minimum zone is controlled by the points P_1, P_2, P_3 and P_4!

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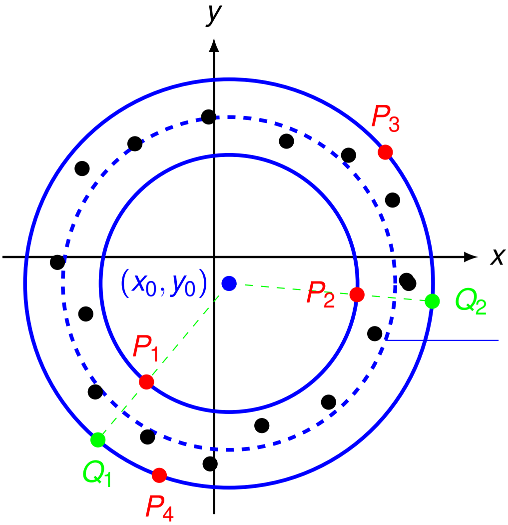

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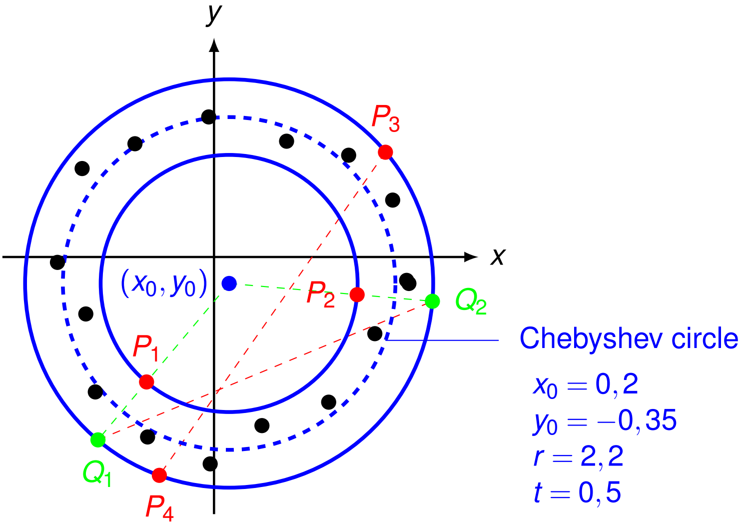

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The second possibility for an optimal solution of the minimum zone circle problem in 2D is:

There are six contacting points.

Example 4: Minimum zone circle in 2D

The second possibility for an optimal solution of the minimum zone circle problem in 2D is:

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- Three contacting point must lie on the inner boundary circle of the zone, the other three on the outer boundary circle.

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- When the three contacting points on the inner boundary circle are radially projected on the outer boundary circle, these projected points must strictly fall on the three contacting points on the outer boundary circle.

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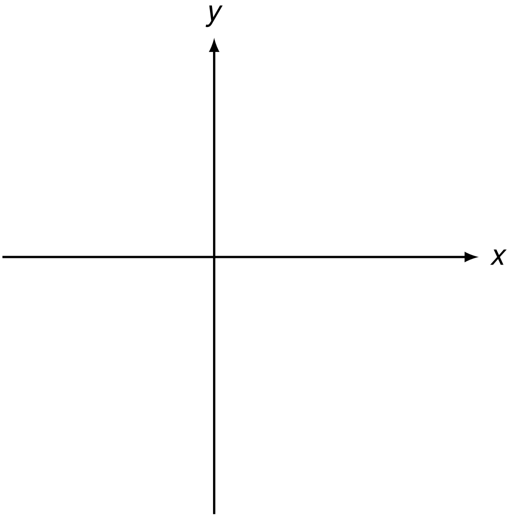

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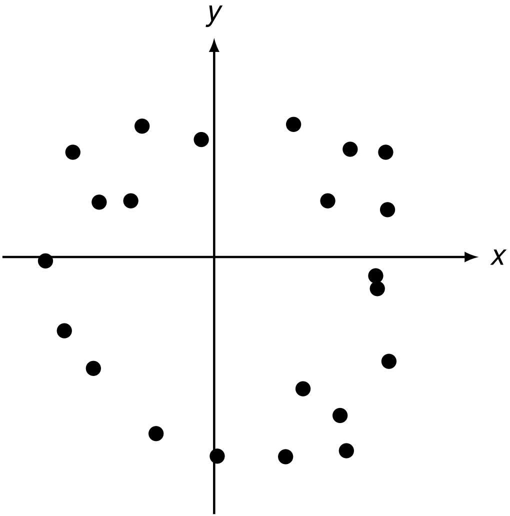

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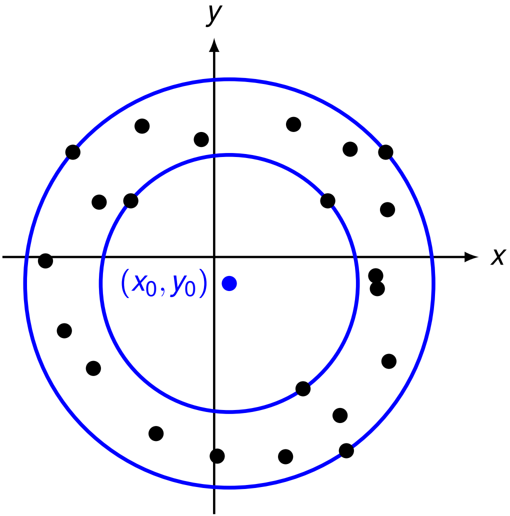

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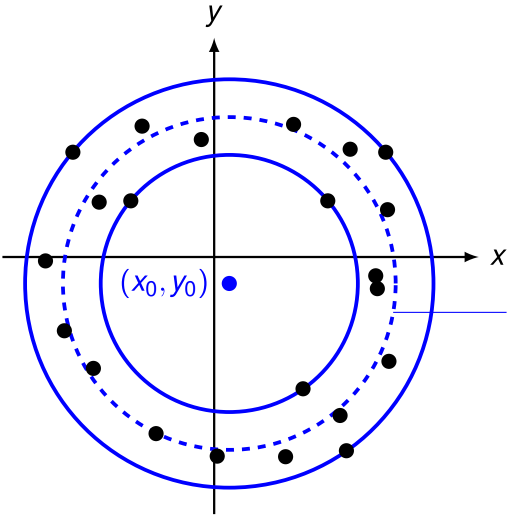

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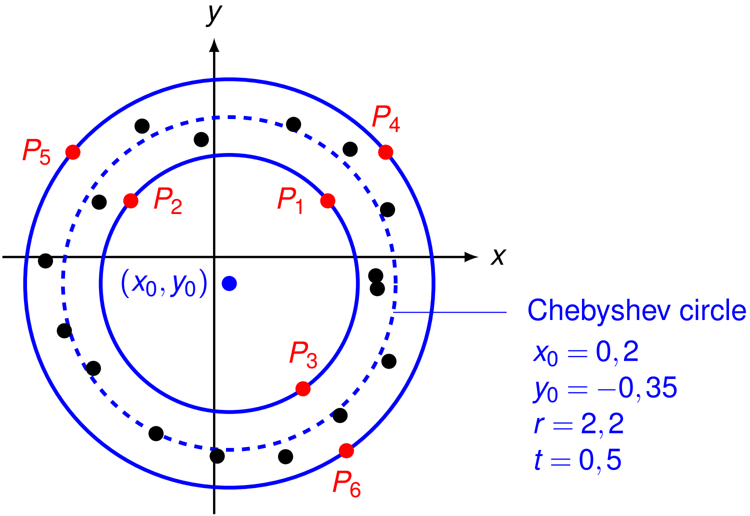

The minimum zone is controlled by the points P_1 to P_6!

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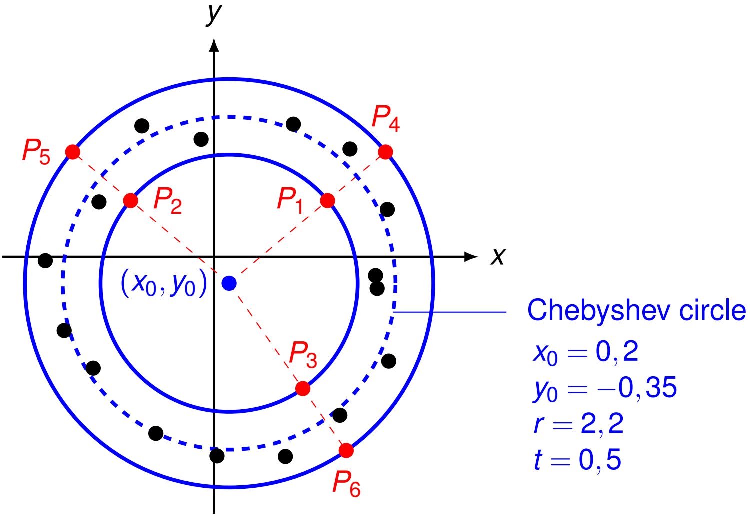

The minimum zone is controlled by the points P_1 to P_6!

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The distance of the i-th measured point (x_i, y_i) from the centre of a circle in 2D is given by

$$d_i = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} > 0,$$

where x_0 and y_0 are the centre coordinates.

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If we set

$$r = \max_{i=1,\ldots,n} d_i(x_0, y_0),$$

we have $d_i \leq r$ ($i = 1, \ldots, n$), i. e. all measured data points are included by a circle of radius r, which is centred at (x_0, y_0).

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$$r > 0 \quad \text{and} \quad r - \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} \geq 0, \quad i = 1, \dots, n.$$

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The cord connecting the two contacting points must pass through the centre of the minimum including circle, i. e. these points must be located on the diameter of this circle.

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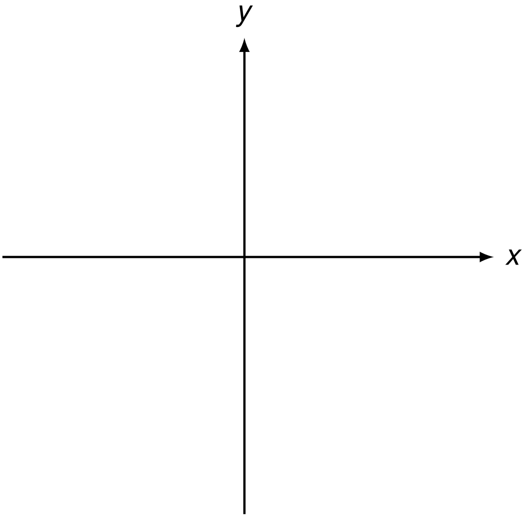

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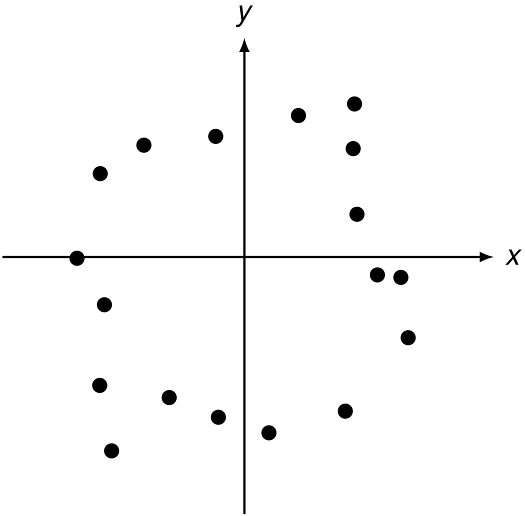

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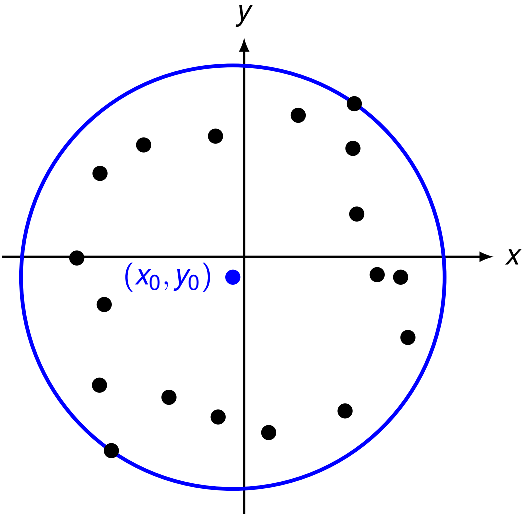

$x_0 = -0,15$
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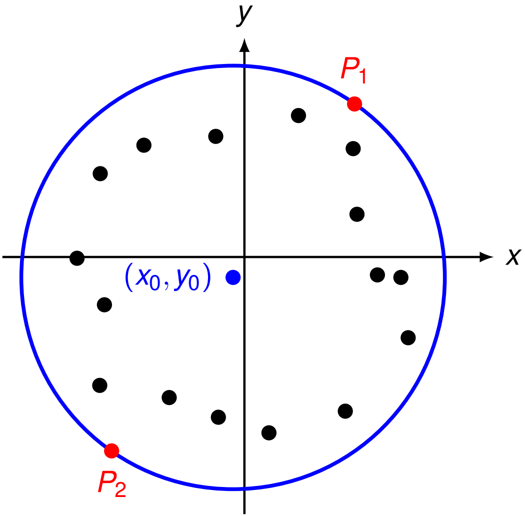

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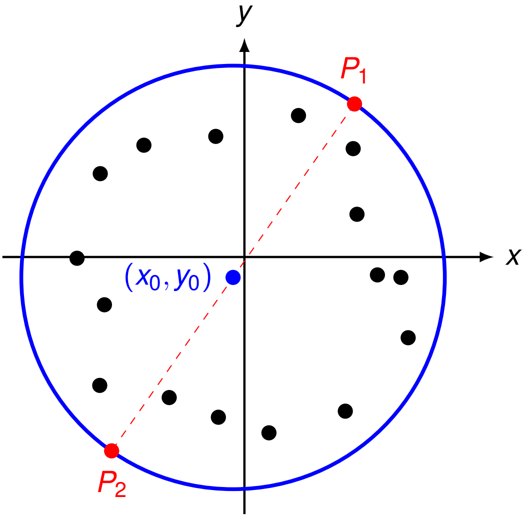

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The three contacting points are the vertices of a strictly acute triangle, which contains the centre of the minimum including circle.

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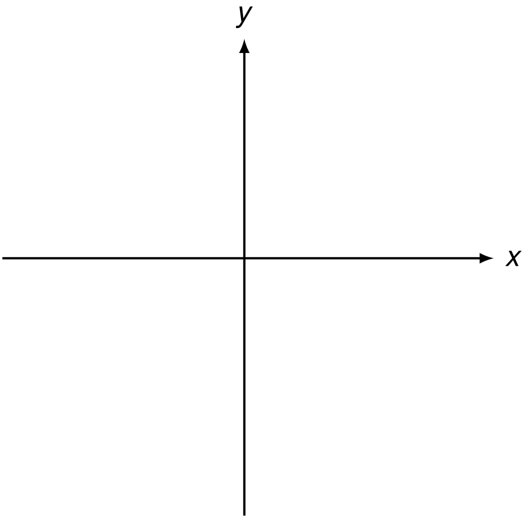

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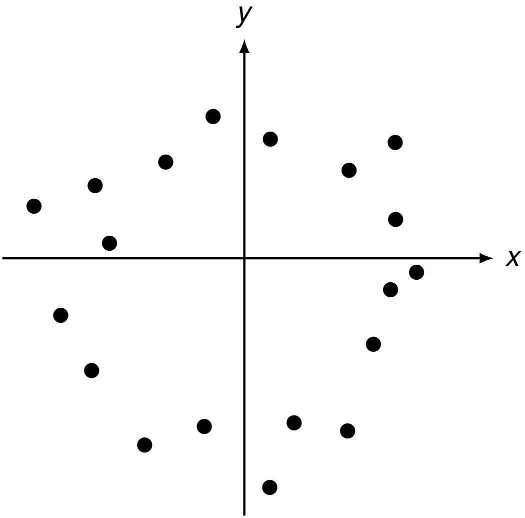

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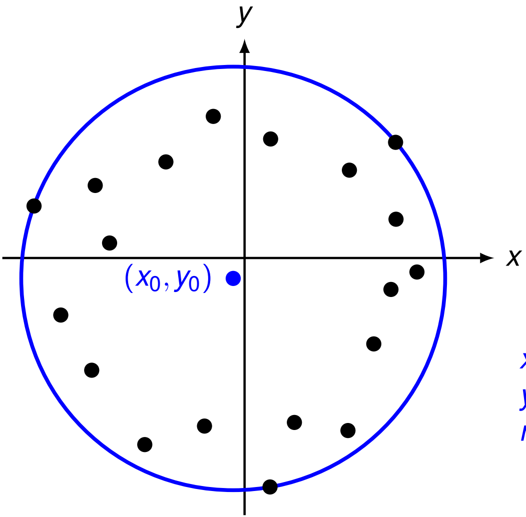

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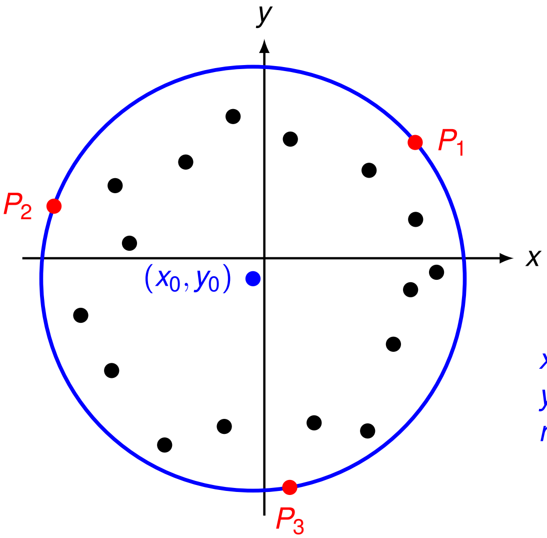

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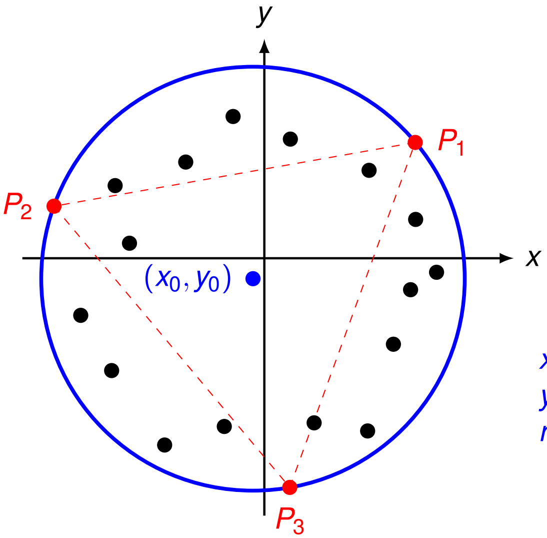

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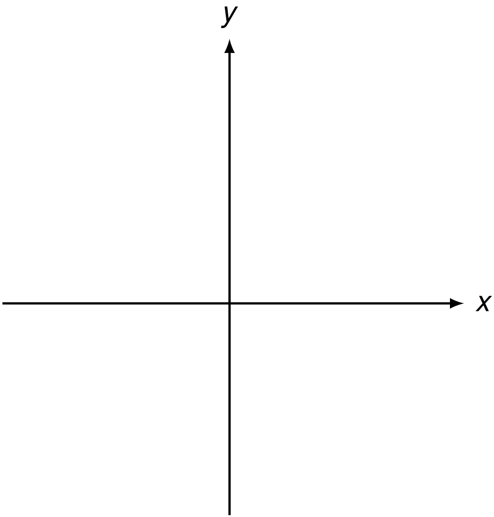

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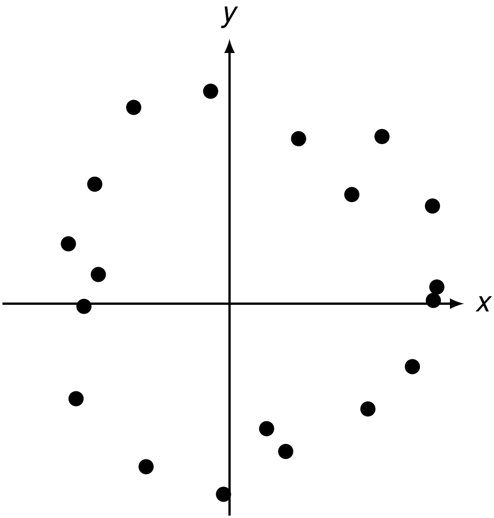

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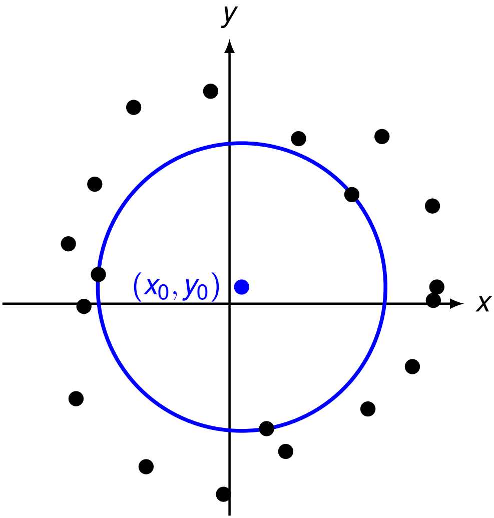

$x_0 = 0,16$
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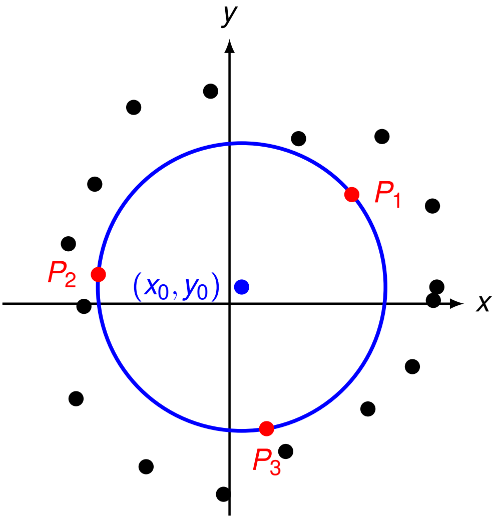

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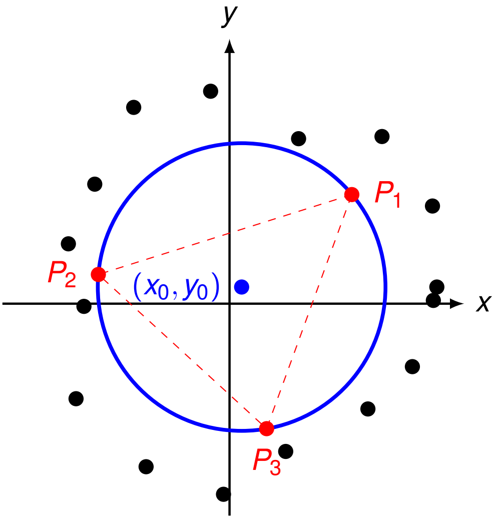

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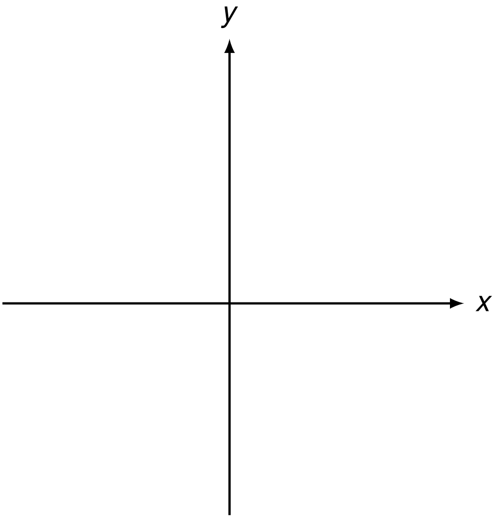

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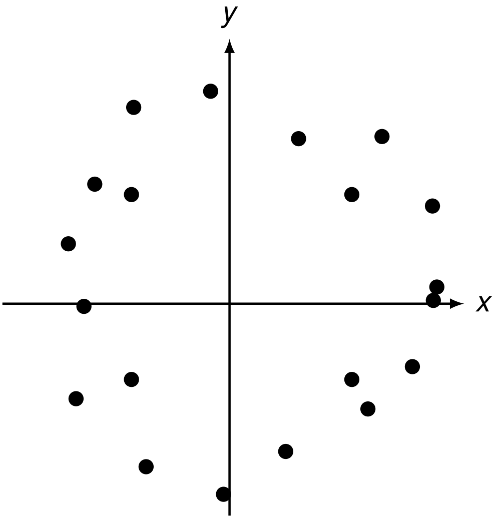

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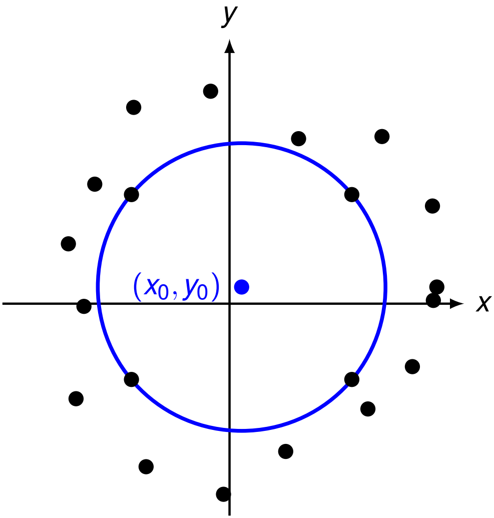

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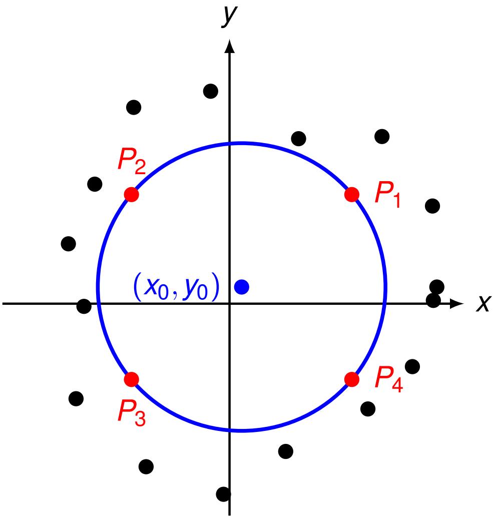

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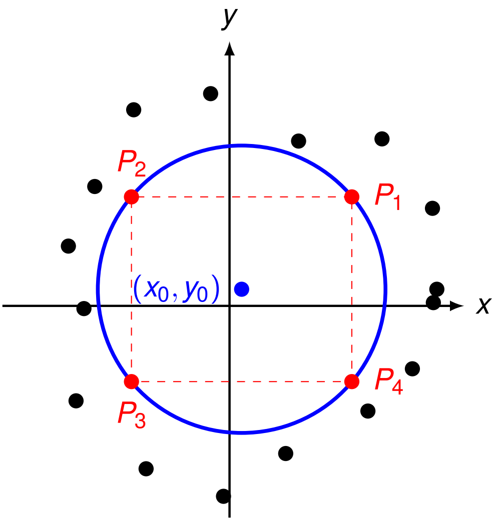

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Final remarks

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For some cases, for example for the least squares straight line in 2D, the least squares plane in 3D, the minimum including circle in 2D, and the minimum including sphere in 3D, there is one unique global optimum and algorithms exist to find it.

In other cases it can not be guaranteed for any algorithm, that it finds the global optimum. However, there are some algorithm, such as the computationally costly combinatorial exhaustive search, which can find all optimal parameter sets and thus also the global optimum (or optima, if no unique solution exists).