

BAYESIAN MEASUREMENT UNCERTAINTY EVALUATION: A COMMENT ABOUT THE GUM SUPPLEMENT 1

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Abstract: Supplement 1 of the GUM provides examples on measurement uncertainty evaluation from the point of view of a Bayesian interpretation of the probability that is stronger than in the GUM itself, but it is not explicit about simulating methods for posterior probability density functions. In this document we point out that situation and present the Metropolis-Hastings simulation technique as an alternative to the sampling method used in the Supplement 1 of the GUM that can be easily applied to simulate posterior distributions. Examples implemented in LabVIEW™ are provided to the reader.

1. INTRODUCTION

Supplement 1 to the GUM [1] provides examples on the evaluation of the measurement uncertainty by the propagation of distributions using a Monte Carlo Method and claims that all those evaluations are in agreement to the Bayesian theory of probability and can be used in practical situations.

But in almost all examples the probability density functions (PDFs) related to the input quantities $g_{X_i}(\xi_i | I)$ are assigned, their parameters are assumed to be known and the use of the likelihood related to the measurements $\{x_i\}$, $l_{X_i}(\{x_i\} | x_i, I)$, is not considered, and that is far from normal practice in the laboratory. There is not information for practitioners for tackling the simulation of posteriori pdfs that are given by the product of a likelihood function and a prior distribution according to the Bayes' theorem.

In this document we explain how to combine a prior PDF with the likelihood through some examples and using a Monte Carlo method called Metropolis-Hastings [2] implemented on LabVIEW™ 8.0 software.

2. BASIC PRINCIPLES

For the determination of the measurement uncertainty of a quantity Y given by a measurement model $Y = f(X_1, \dots, X_N)$, the GUM Supplement 1 proposes to evaluate the standard measurement uncertainty of Y from

$$\left[\int_{-\infty}^{\infty} (\eta - y)^2 g_Y(\eta | I) d\eta \right]^{1/2} \quad (1)$$

where $g_Y(\eta)$ is the PDF of Y . On the other hand $g_Y(\eta)$ is evaluated from the joint distribution of Y, X_1, \dots, X_N :

$$g_Y(\eta) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_{Y, X_1, \dots, X_N}(\eta, \xi_1, \dots, \xi_N) d\xi_1 \dots d\xi_N \quad (2),$$

or after applying the multiplicative property of probabilities, $p(AB) = p(A|B)p(B)$:

$$g_Y(\eta) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_{Y|X_1, \dots, X_N}(\eta | \xi_1, \dots, \xi_N) g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N) d\xi_1 \dots d\xi_N \quad (3)$$

If the condition $Y - f(X_1, \dots, X_N) = 0$ is imposed to equation (3) by means of Dirac's Delta distribution, δ , then it becomes in the expression recommended in section 5.2, (the "Markov formula"):

$$g_Y(\eta) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta(\eta - f(\xi_1, \dots, \xi_N)) g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N) d\xi_1 \dots d\xi_N \quad (4)$$

In particular, if all X_i are independent, equation (4) can be written as follows:

$$g_Y(\eta | I) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta(\eta - f(\xi_1, \dots, \xi_N)) \prod_{i=1}^N g_{X_i}(\xi_i | I) d\xi_1 \dots d\xi_N \quad (5)$$

The evaluation of (4) can be done analytically for some simple measurement models with two or three variables, but for those models which are used in everyday applications or calibration laboratories, to find an analytical solution could be either very difficult or not possible and it is necessary to use Monte Carlo Methods that comprise a long variety of simulation techniques. In particular, the Supplement 1 of the GUM [1] proposes sampling from the inverse of the cumulative distribution in its Annex C.

3. THE SUBJECT OF THIS PAPER

According to section 6.2 of [1] the PDF $g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N)$ can be evaluated using the Bayes' theorem:

$$g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N | \{x_1\}, \dots, \{x_N\}, I) \alpha \int_{X_1, \dots, X_N} l_{X_1, \dots, X_N}(\{x_1\}, \dots, \{x_N\} | x_1, \dots, x_N, I) g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N | I) \quad (6)$$

where I was added in order to represent any additional information available. Then equation (6) can be re-written as:

$$g_Y(\eta | \{x_1\}, \dots, \{x_N\}, I) \alpha \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta(\eta - f(\xi_1, \dots, \xi_N)) l_{X_1, \dots, X_N}(\{x_1\}, \dots, \{x_N\} | x_1, \dots, x_N, I) g_{X_1, \dots, X_N}(\xi_1, \dots, \xi_N | I) d\xi_1 \dots d\xi_N \quad (7)$$

And the normalization factor K is given by

$$K = \left[\int_{-\infty}^{\infty} g_Y(\eta | \{x_1\}, \dots, \{x_N\}, I) dy \right]^{-1} \quad (8)$$

Equations (6) and (7) are not explicit provided in the GUM Supplement 1, but are a clear consequence of the Bayes' theorem.

In the GUM Supplement 1 the PDFs of the input quantities $g_{X_i}(\xi_i | I)$ are assigned and their parameters are assumed to be known exactly; the examples do not consider the use of the likelihood related to the measurements $l_{X_i}(\{x_i\} | x_i, I)$, and [1] does not explain how to evaluate equation (6), which naturally arise when measurement data is also available.

The only exception is when the t-distribution is used in example 9.5 in [1]. In that case, the t-distribution arises when the likelihood or sampling distribution is assumed to be normal, and non informative priors are assigned to the mean and variance of the quantity of interest. Also the t-distribution and the gamma distribution are mentioned in section 6.4 of [1] and their origins are explained in terms of the same likelihoods and priors.

Probably this is due to the fact that it is not so immediate to find the cumulative distribution inverses for all posterior PDFs, by drawing samples from them. But there is a more versatile simulation technique that could be considered by metrology practitioners and that can handle this type of situations: the Metropolis-Hastings (M-H) algorithm.

4. METROPOLIS-HASTINGS ALGORITHM

The Metropolis-Hastings (M-H) algorithm appeared on 1953 [3] as a mathematical tool for the calculation of the solution of a problem related to statistical mechanics and many years latter statisticians found it to be very useful for the simulation of complicated probability distributions. Although advanced mathematical concepts like "Markov Chains" [2, 4] support the M-H simulation algorithm, its application is pretty simple in many cases and the M-H algorithm need only the conditions d) and e) stated in 5.10 of [1].

For the simulation of a PDF $g_X(\xi)$ it is necessary to specify a generating-jumping rule by a generating distribution $q_{X_{t+1}|X}(\xi_{t+1} | \xi_t)$ and a step size d . This rule will provide information about how to generate a new value ξ_{t+1} when the previous value was ξ_t . Large d values imply faster to get simulations (in the sense that the simulation region is quickly covered) lower accuracy, and sometimes non-convergence of the simulation to the desired probability distribution, $g_X(\xi)$. A good selection of d values depends on the relative scale of the problem and experimentation; this is a heuristic process.

In this document we will use the uniform distribution $U(0,1)$, for the definition of the generating-jumping rule so it's easy to compute and it may be available in many software. Then, the generation and jumping rule is defined as:

$$\begin{cases} \text{if } U(0,1) < 0,5 \Rightarrow \xi_{t+1} = \xi_t + d \\ \text{if } U(0,1) \geq 0,5 \Rightarrow \xi_{t+1} = \xi_t - d \end{cases} \quad (9)$$

thus, the M-H algorithm is as follows:

1. Set $t=0$ and an arbitrary value $\xi_{t=0}$
2. Generate a new value ξ_{t+1} from $q_{X_{t+1}|X_t}(\xi_{t+1} | \xi_t)$ (when $t=0$ the new value is ξ_1 and is obtained from $q_{X_1|X_0}(\xi_1 | \xi_0)$); if $t=1$, ξ_2 comes from $q_{X_2|X_1}(\xi_2 | \xi_1)$, etc.)
3. Decide to accept or reject the new value ξ_{t+1} according to the following rule

$$\begin{cases} \text{if } U(0,1) \leq r \Rightarrow \xi_{t+1} \equiv \xi_{t+1} \\ \text{if } U(0,1) > r \Rightarrow \xi_{t+1} \equiv \xi_t \end{cases} \quad (10)$$

Where,

$$r = \frac{g_X(\xi_{t+1})q_{X_t|X_{t+1}}(\xi_t | \xi_{t+1})}{g_X(\xi_t)q_{X_{t+1}|X_t}(\xi_{t+1} | \xi_t)} \quad (11)$$

In the case of the uniform distribution we have

$$q_{X_t|X_{t+1}}(\xi_t | \xi_{t+1}) = q_{X_{t+1}|X_t}(\xi_{t+1} | \xi_t), \text{ hence,}$$

$$r = \frac{g_X(\xi_{t+1})}{g_X(\xi_t)} \quad (12)$$

4. Go back to step 2 in order to generate a new value from the last one and repeat this process M times up to reach the convergence for $g_X(\xi)K$ where K is the frequency distribution of the sampled values ξ_t (see Eq. 8). This might take many iterations.

The first example of M-H implementation is presented together with the **normaldistribution.vi** file; the routine implemented in this file simulates a normal distribution with known variance and mean. It is presented for a short heuristic discussion and to providing a general picture of the M-H method, it does not pretend neither to evaluate its convergence nor other properties. The whole files referred into this paper can be downloaded from **simulations.rar** [5] (the password is 4628) and they are implemented in LabVIEW™ [6]. This software (version 8.0), has been chosen so because it is frequently available in many calibration laboratories and it is easy to apply without any previous

programming experience. LabVIEW™ programming interface is visual and it looks like a flow chart. It also provides a useful interface called “Front panel” that can be configured by the user. Executable versions are available in the **simulations.rar** file. It is necessary previously to install the Run Time Engine 8.0 [7].

In Figure 1, a simulation of the Normal distribution with mean = 12 and variance = 1 $N(12, 1)$, is presented. The visual code “block diagram”, of **normaldistribution.vi**, is presented. The simulated numbers are generated according to the previous algorithm in the upper half of the block diagram. The lower half of the block diagram is related to the presentation of the results: the evaluation of the mean, variance, and the graphical representation (*histogram, waveform chart and cumulative distribution*).

In Figure 2, the simulation results are found by running 3012 iterations. The mean of the simulated values is 11,14 and the variance is 8,32, the step size is 0,1 and the initial value is 0. This is possible to see in the *waveform chart* graph how the simulation evolves from 0 to a value close to the mean. In this case the number of iterations is not enough for obtaining the desired PDF, i.e. the shape of the *cumulative distribution* graph is quite far away from the characteristic “s” shape of the normal distribution. The *histogram* graph shows a bi-modal distribution and the values from the left correspond to the first iterations. In Figure 3, 65000 iterations are obtained from the same file, the mean of the simulated values is 11,96 and the variance is 1,25. The *histogram* and *cumulative distribution* graphs present the typical shape of the normal distribution. In the wave form chart we can see how the last iterations move around 12.

In Figure 4 another simulation of $N(12, 1)$ with 10000 values is presented. Now, the step size is 1 and the convergence was faster. The mean of simulated values is 12,01 and the variance is 1,03. This is possible to see in the *waveform chart* graph that discrete values are simulated and the distance between them is determined by the step size.

In this example, is easy to realize that this is not necessary to know the inverse of the cumulative density function when using the M-H algorithm; it is enough writing the analytic expression of the PDF. Its application is easy to extend for PDFs bigger than one dimension.

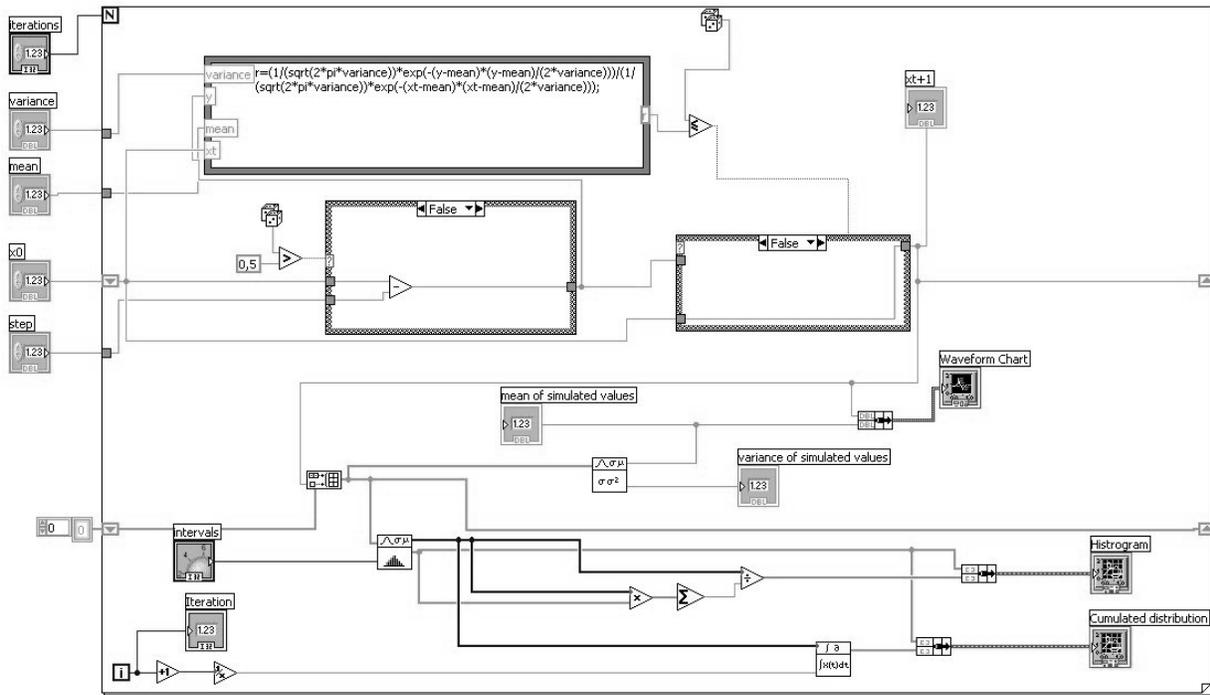


Fig. 1 Block diagram of normaldistribution.vi

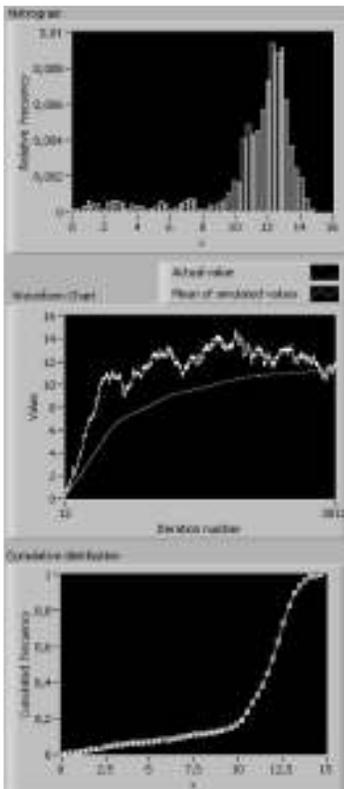


Fig. 2 Normal distribution simulation. 3012 iterations. Step size equal to 0,1

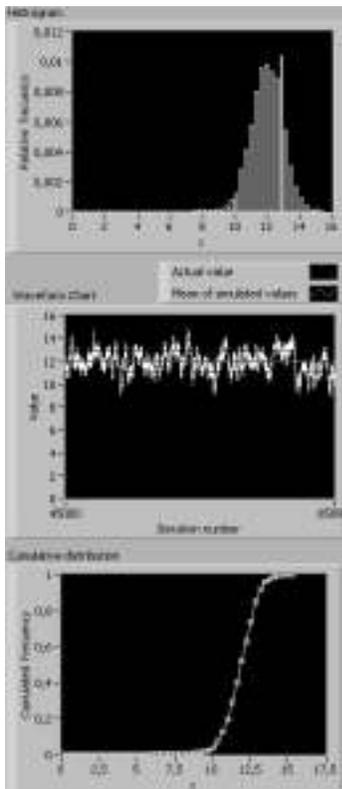


Fig. 3 Normal distribution simulation. 65000 iterations. Step size equal to 0,1

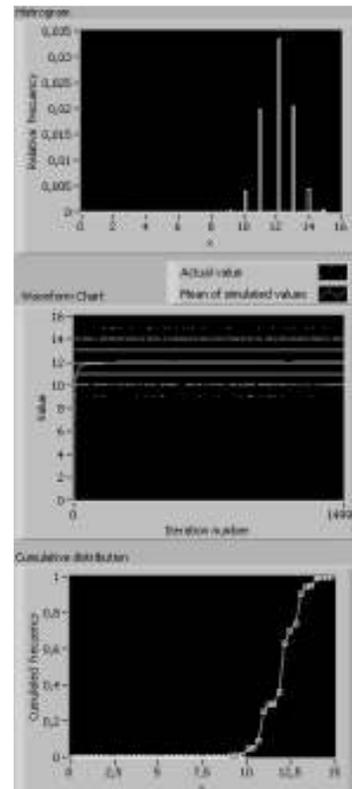


Fig. 4 Normal distribution simulation. 10000 iterations. Step size equal to 1

This is necessary to have in mind that the searching convergence process from the simulation values may tend to stop suddenly or unexpectedly around certain values other than the true value(s), specially in the simulation of PDFs with more than one maximum, for example during the simulation of a bivariate normal distribution.

In order to avoid that problem, a tempering simulation technique based on M-H can be applied by a pumping technique [8]. Even though this is outside of the interests of this paper an example is provided to the reader from the **simulacionbivariadatempering.vi** file, which corresponds to the simulation of the sum of two bivariate normal PDFs and works with the **sub-vi pasosimulacionmultinormal.vi** file. Correlated and uncorrelated components are considered for each variable.

The concept of randomness, which is not Bayesian, has been created to deal with a lack of information or ignorance from the point of view of the frequentist concept of probability, e.g.: “random error”. Even though LabVIEW™ applies the Wichmann-Hill generator for the uniform distribution, described in [9] and superseded many years later [10], its cycle length (~10¹²) is still good enough for our purposes and since true random generators have not been applied yet, they seem to behave randomly.

Reference [1] supports the use of the Bayes theorem, but surprisingly mentions the concept of randomness along its pages, particularly in section 3, “Terms and definitions”. Far from considering that situation as a problem or error, the authors’ opinion is that this inconsistency, widely founded in recognized references, is a symptom of the evolution of the nature of the concepts in metrology. In the original version of 1995 [11] the Bayesian concepts were latent but in [1] they are mentioned along many sections of the document; this shows that the metrology community is moving towards the Bayesian interpretation of probability.

5. POSTERIOR SIMULATION WITH M-H

A posterior PDF is given by the product of a likelihood function and a prior PDF and its simulation by M-H is done in the same way as in previous examples so it is just a PDF. The only difference is that it is necessary to define the prior PDF for each case according to the available information and sometimes measurements are available for defining

the likelihood function. The prior PDF assignment can be done by applying the principle of maximum entropy (PME) as mentioned in 6.3 of [1]. The following examples illustrate this idea. Because of space limitations it is not possible to provide a large discussion for all examples, but their implementations are available in the **vi** files for the reader. We concentrate our discussion in the first and fourth example.

5.1. First example

As a first example we will consider the case where the only prior information available for the quantities X (location parameter) and Σ^2 (scale parameter) is that their values are within the intervals $[a, b]$ and $[c, d]$ respectively ($c, d \geq 0$). A set of normally distributed observations is available, $l_X(\{x\} | \xi, \sigma^2, I)$, and non-informative priors are assigned to X and Σ^2 . X and Σ^2 are independent with prior distributions represented by $g_{X|I}(\xi | I)$ and $g_{\Sigma^2|I}(\sigma^2 | I)$. From the Bayes theorem (see equation (6)) and the PME is obtained:

$$g_{X, \Sigma^2 | \{x\}, I}(\xi, \sigma^2 | \{x\}, I) \alpha \quad (13)$$

$$l_X(\{x\} | \xi, \sigma^2, I) g_{X|I}(\xi | I) g_{\Sigma^2|I}(\sigma^2 | I)$$

$$g_{X, \Sigma^2 | \{x\}, I}(\xi, \sigma^2 | \{x\}, I) \alpha \quad (14)$$

$$\frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2} \sum_1^n (x_i - \xi)^2} \frac{1}{b-a} (H(\xi - a) - H(\xi - b)) \times$$

$$\times \frac{1}{\sigma^2 \ln(d/c)} (H(\sigma^2 - c) - H(\sigma^2 - d))$$

$H(\xi)$ is the Heaviside Step Function [12], which is very useful for the construction of polygonal shape distributions, in this case, a uniform distribution. The factors $\frac{1}{b-a}$ and $\frac{1}{\ln(d/c)}$ can be eliminated, so these are only normalization constants of $g_{X|I}(\xi | I)$ and $g_{\Sigma^2|I}(\sigma^2 | I)$ on the other hand cancel in the evaluation of r , equations (11 and 12).

The sum $\sum_1^n (x_i - \xi)^2$ can be re-written as $(n-1)s^2 + n(\bar{x} - \xi)^2$, where

$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$ and $\bar{x} = \frac{1}{n} \sum x_i$ are the variance and mean of the observations.

Thus:

$$g_{X, \Sigma^2 | \{x\}, I}(\xi, \sigma^2 | \{x\}, I) \propto \frac{1}{\sigma^{n+2}} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x} - \xi)^2)} (H(\xi - a) - H(\xi - b)) (H(\sigma^2 - c) - H(\sigma^2 - d)) \quad (15)$$

The file **sim1.vi** provides a simulation implemented using M-H. In Figures 5 and 6 show simulations for X and Σ using $a=0, b=12, c=0,1, d=5, \bar{x}=13,2, s^2=1,70$ and $n=5$ (The values of \bar{x} and s^2 can be

obtained from the set of numbers {12, 12, 13, 14, 15}). The initial value for the mean is $\xi_0 = 6$ and the initial value for the variance is $\sigma_0^2 = 3$, the step size is 0,1 for ξ and σ .

For metrology purposes we are interested in the simulation of X (Figure 5) and its standard deviation which is the standard uncertainty; Σ is a nuisance parameter (Figure 6).

Table 1 shows how the range of variation of the simulation results of the parameters of interest get smaller with a larger iteration number that may change according to the step size.

Table 1. Results of 20 trials with different number of iterations

Number of iterations	Number of trials	Mean of the trials means	Mean of the trials standard deviations	Range of variation of trials means	Range of variation of trials standard deviations
10000	5	11,44	0,94	0,37	0,27
30000	5	11,51	0,70	0,10	0,19
100000	5	11,56	0,63	0,09	0,09
300000	5	11,61	0,49	0,02	0,07

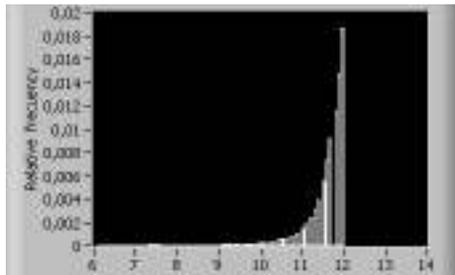


Fig. 5 Simulation of the posterior pdf of X

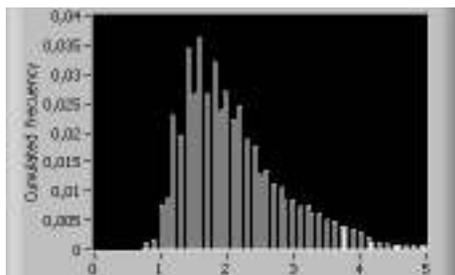


Fig. 6 Simulation of the posterior pdf of Σ

5.2. Second example

In this example the prior information for X says that is located in the interval $(0, \infty)$ and it's best estimate is equal to μ_p . Σ^2 is located in the interval $[c, d]$.

A set of normally distributed observations are available, $l_X(\{x\} | \xi, \sigma^2, I)$. A non-informative prior is assigned to Σ^2 . From the Bayes theorem and the PME:

$$g_{X, \Sigma^2 | \{x\}, I}(\xi, \sigma^2 | \{x\}, I) \propto \frac{1}{\sigma^{n+2}} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x} - \xi)^2)} \times H(\xi) e^{-\frac{\xi}{\mu_p}} (H(\sigma^2 - c) - H(\sigma^2 - d)) \quad (16)$$

See file **sim2.vi** for the implementation.

5.3 Third example

In this example we consider a model with two variables: $Y = X_1 + X_2$. Simulated values for each variable are drawn using M-H at the same time. Then each pair of values is used to evaluate the measurement model as many times as defined by the user and a histogram is constructed with the results. In the file is *sim3.vi* the posterior PDF of the first example was assigned to each variable; in *sim4.vi* the posterior PDF of the first example was assigned to X_1 and the posterior of the second example was assigned to X_2 .

5.4. Fourth example

As suggested by one referee we included an example using a measurement model more complicated than the previous one in order to show an application closer to metrology practitioners' reality. In this case we choose the measurement model for the air density, ρ_a , proposed in equation (E.3-1) of [13], and we will consider the information related to measurement instruments calibration results and resolution. The vi file is *airdensity.vi* and requires the use of the *graphics.vi* file as a subroutine.

The measurement model for ρ_a (in kg/m^3) is:

$$\rho_a = \frac{0,0034848(P + \delta D_p + C_p)}{273,15 + (T + \delta D_T + C_T)} + \frac{0,009(H + \delta C_H + C_H) \times e^{0,0612(T + \delta D_T + C_T)}}{273,15 + (T + \delta D_T + C_T)} + \delta \rho_f \quad (17)$$

Where,

P is the atmospheric pressure [Pa],

T is the temperature [°C], and

H is the relative humidity [%].

P , T , and H are determined by measurements and prior information. The posterior PDF of the first example is assigned to each one. These are the only variables with associated likelihoods.

δD_p , δD_H , and δD_T are corrections for compensating the error due to the finite resolutions of the instruments used for the measurements of pressure, humidity, and temperature, respectively. A uniform distribution, centered at zero, is assigned to each one. The width of each uniform distribution is equal to the resolution of the instrument.

C_p , C_H , and C_T are the corrections reported in the calibration certificates of the instruments used for the measurements of pressure, humidity, and temperature, respectively. This is assumed that the condition of clause 6.4.9.8 of GUM Supplement 1 [1] applies, i.e. normal distributions are used for C_p , C_H , and C_T . $\delta \rho_f$ is the correction for compensating the error of the air density equation. Its mean value is equal to 0 and its standard uncertainty is equal to $1 \times 10^{-4} \rho_a$; for the standard uncertainty we will make the approximation $1 \times 10^{-4} \text{ kg/m}^3$ so ρ_a is a simulated value. In OIML R111-1:2004 [13] there is not information about the probability density function of $\delta \rho_f$, but we will assume it to be normal, we know its mean value (0) and its standard uncertainty.

Figure 7 shows the histogram of 50000 simulations done with the conditions described in Table 2, which are the default conditions that the reader will find when opening the vi file (except the values for the likelihood function that have to be introduced by hand). The PDF of the density is skewed because the observed temperature values are outside of the temperature prior PDF (smaller). The mean value obtained was $1,129\ 89 \text{ kg/m}^3$ with a standard uncertainty equal to $0,000\ 85 \text{ kg/m}^3$

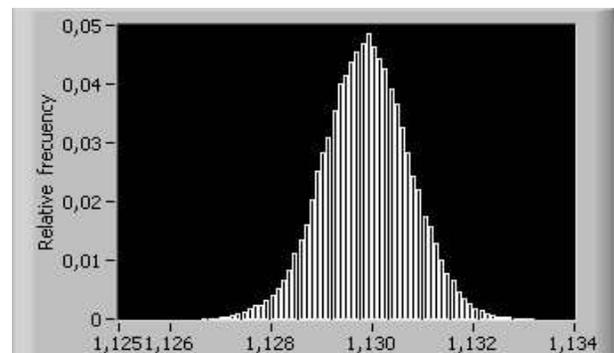


Fig. 7 Simulation of the posterior pdf of the air density.

The posterior PDF resembles the normal distribution due to the contributions of the many variables that participate in this model and the central limit theorem. But it is not normal according to a Kolmogorov-Smirnov test [14] done on the last 48000 simulated values (the first 2000 were not considered so the first simulated values oscillate more than the last); it is little skewed to the right.

The posterior PDF of T contributes to this lack of symmetry so it is not symmetric and the temperature has a high effect on the air density. The posterior

PDF of T is not symmetric because the values used to construct the likelihood are higher or equal to the lowest values of the prior PDF of T . From the cumulative frequency of the simulated values it is possible to determine a coverage interval with a coverage probability equal to 95%, for

example using as limits the 2,5% and 97,5% percentiles, we get [1,1282 , 1,1316]. In order to assure the convergence it would be possible to consider the evaluation of the results of running simultaneous simulations.

Table 2. Simulation conditions of the posterior PDF of the air density

Variable/ Unit	Likelihood	Measured values	Prior PDF of the mean	Prior PDF of the variance	Initial value of the mean simulation / step	Initial value of the variance simulation / step
$T / ^\circ\text{C}$	Normal	20,5; 20,5;20,9	U(20,00 , 20,50)	$1/(\sigma^2 \ln(5/0,1))$	20,5 / 0,05	3 / 0,05
$H / \text{hr}\%$	Normal	46; 48;49	U(45, 50)	$1/(\sigma^2 \ln(5/0,1))$	47 / 0,1	3 / 0,1
P / Pa	Normal	95100; 95150	U(94000, 98000)	$1/(\sigma^2 \ln(10^4/0,1))$	95100 / 5	100 / 10
$\delta D_T / ^\circ\text{C}$	--	--	U(-0,05, 0,05)	--	-0,05 / no applies	--
$\delta D_H / \text{hr}\%$	--	--	U(-0,05, 0,05)	--	-0,05 / no applies	--
$\delta D_P / \text{Pa}$	--	--	U(-50, 50)	--	-50 / no applies	--
$C_T / ^\circ\text{C}$	--	--	N(0, 0,05 ²)	--	0,01 / 0,01	--
$C_H / \text{hr}\%$	--	--	N(0, 1 ²)	--	0,1 / 0,01	--
C_P / Pa	--	--	N(45, 50 ²)	--	0,02 / 3	--
$\delta \rho_f / \text{kg/m}^3$	--	--	N(0, 0,001 ²)	--	0 / 0,00002	--

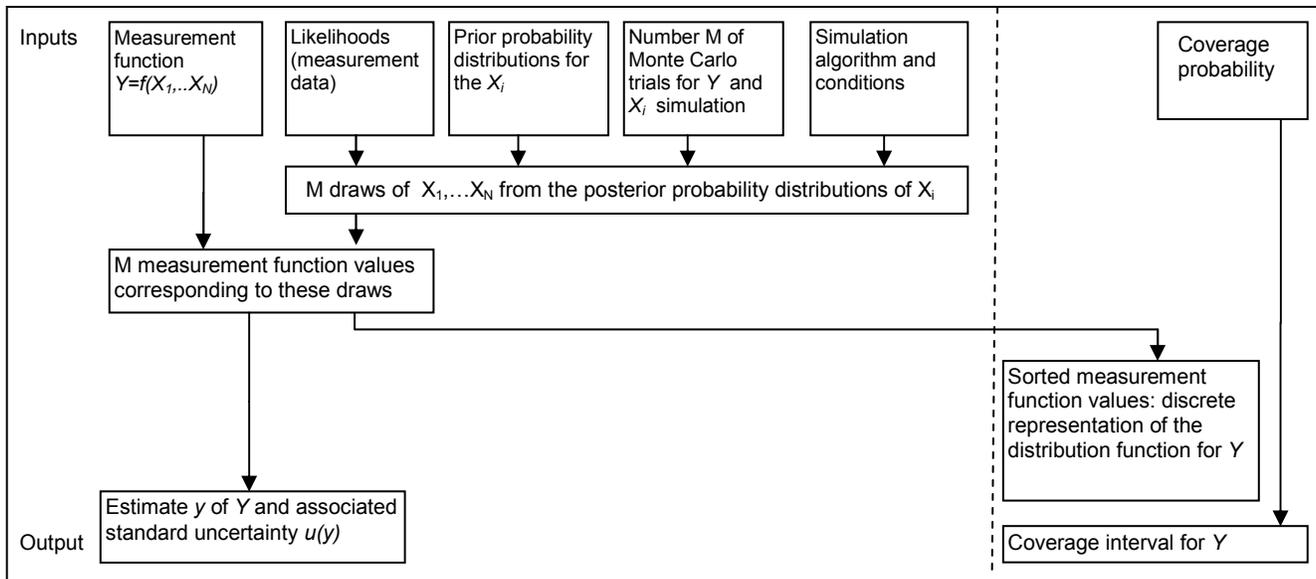


Fig. 8 Reformulated scheme for the Monte Carlo method.

6. FINAL REMARKS

JCGM 104:2009 (Figure 8) [11] gives a representation of the measurement uncertainty evaluation procedure using a Monte Carlo Method that might be improved by the scheme from Figure 8

which shows an improved version taking on account the concepts already discussed.

7. CONCLUSIONS

In this paper the M-H algorithm was presented as an alternative to the Monte Carlo Method proposed in the Supplement 1 of GUM which is based on sampling from the cumulative PDF.

The M-H algorithm was used to explain how to simulate PDFs, particularly posterior PDFs (given a prior PDF and a likelihood functions), and it might be useful by the metrology community.

Further research is needed to determine the optimum simulation conditions (step sizes, initial values, number of iterations, among others) for each measurement model.

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